

The local C^1 -density of stable ergodicity

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Abstract

The center bundle of a conservative partially hyperbolic diffeomorphism f is called robustly non-hyperbolic if any conservative diffeomorphism which is C^1 -close to f has non-hyperbolic center bundle. In this paper, we prove that stable ergodicity is C^1 -dense among conservative partially hyperbolic systems with robust non-hyperbolic center.

Keywords: partial hyperbolicity; stable ergodicity; Lyapunov exponents; blender

1 Introduction

Let M be a smooth compact, connected and boundless Riemannian manifold with dimension $d \geq 3$, and μ be a smooth volume measure on M with $\mu(M) = 1$. Denote by $\text{Diff}_\mu^r(M)$ the set of C^r μ -preserving diffeomorphisms of M endowed with C^r topology for $r \geq 1$. If $f \in \text{Diff}_\mu^r(M)$, we also call f is a conservative system.

A diffeomorphism $f : M \rightarrow M$ is said to be *partially hyperbolic*, if f admits a non-trivial Df -invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$ and numbers $0 < \alpha_s < \alpha'_c \leq \alpha''_c < \alpha_u$ such that $\alpha_s < 1 < \alpha_u$ and for any $x \in M$, we have

$$\|Df|_{E^s(x)}\| < \alpha_s, \quad \alpha'_c \leq m(Df|_{E^c(x)}), \quad \|Df|_{E^c(x)}\| \leq \alpha''_c \quad \text{and} \quad \alpha_u < m(Df|_{E^u(x)}),$$

where $m(Df|_E)$ is the minimum norm of $Df|_E$, i.e.,

$$m(Df|_E) = \inf\{\|Df v\| : v \in E, \|v\| = 1\}.$$

The subbundles E^u, E^c and E^s are called *unstable*, *center* and *stable bundle*. Set $\beta = \dim(E^\beta)$ for $\beta = s, c, u$. Partial hyperbolicity is a robust property. That is to say, for any given partially hyperbolic diffeomorphism f of M , there is a C^1 neighborhood \mathcal{U} of f in $\text{Diff}^1(M)$ such that any $g \in \mathcal{U}$ is partially hyperbolic. We denote by $\text{PH}_\mu^r(M)$ the family of C^r conservative partially hyperbolic diffeomorphisms of M endowed with C^r topology for $r \geq 1$. Given $f \in \text{PH}_\mu^1(M)$, the center bundle E_f^c of f is called *robustly non-hyperbolic* if there is a C^1 neighborhood \mathcal{U} of f in $\text{PH}_\mu^1(M)$ such that each $g \in \mathcal{U}$ has two ergodic measures μ_1 and μ_2 satisfy $\lambda_{\mu_1}^+ \leq 0$ and $\lambda_{\mu_2}^- \geq 0$, where $\lambda_{\mu_1}^+$ and $\lambda_{\mu_2}^-$ are the largest and smallest Lyapunov exponents of μ_1 and μ_2 in E_g^c .

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We set

$$\mathcal{P} = \{f \in \text{PH}_\mu^1(M) : E_f^c \text{ is robustly non-hyperbolic}\}.$$

Then \mathcal{P} is a non-empty open subset of $\text{PH}_\mu^1(M)$. The openness is obvious by the definition. On the other hand, if a conservative partially hyperbolic system f have two hyperbolic periodic points with indices s and $s + c$ respectively, then $f \in \mathcal{P}$. This implies that \mathcal{P} is non-empty.

A diffeomorphism $f \in \text{Diff}_\mu^1(M)$ is *ergodic* (with respect to μ) if only full or null μ -volume sets are invariant under it. f is *stably ergodic* if there exists a C^1 open neighborhood \mathcal{U} of f in $\text{Diff}_\mu^1(M)$ such that any diffeomorphism $g \in \mathcal{U} \cap \text{Diff}_\mu^2(M)$ is ergodic with respect to μ .

The main result of this paper is

Theorem A. *There is a subset \mathcal{D} of \mathcal{P} such that \mathcal{D} is C^1 -dense in \mathcal{P} and each $f \in \mathcal{D}$ is stable ergodic.*

The study of stable ergodicity has a long-time history. In [2, 3], by using Hopf Argument ([22]), D. Anosov and J. Sinai established ergodicity of all C^2 volume-preserving uniformly hyperbolic systems (Anosov systems), including geodesic flows for compact manifolds of negative sectional curvature. In 1994, M. Grayson, C. Pugh and M. Shub ([19]) gave the first nonuniformly hyperbolic example of a stably ergodic system. These systems are partially hyperbolic. Following this direction, Pugh and Shub believe that a little hyperbolicity goes a long way in guaranteeing ergodicity and, in [26, 25], they posed the following Stable Ergodicity Conjecture:

Conjecture. *Stable ergodicity is C^r -dense among conservative partially hyperbolic diffeomorphisms.*

At the same time, Pugh and Shub gave a program to deal with this conjecture: they conjectured that stable accessibility is dense and essential accessibility implies ergodicity among volume-preserving, partially hyperbolic diffeomorphisms. In recent years, many advances have been made for this conjecture (e.g. see the survey [16, 27]). For example, F. Rodriguez Hertz, M. Rodriguez Hertz and R. Ures ([28]) proved that stable ergodicity is C^∞ -dense among partially hyperbolic diffeomorphisms with one-dimensional center bundle; K. Burns and A. Wilkinson ([17]) proved that essential accessibility implies ergodicity if the system is center bunched, and C. Bonatti, C. Matheus, M. Viana, and A. Wilkinson ([13]) proved the conjecture in the C^1 topology for one-dimensional center bundle.

As pointed in [31, 32], many arguments of previous works (such as [17] and [28]) seem to be hard to generalize and have reached their limits in these directions. Recently, a new alternate criterion to establish ergodicity be obtained by F. Rodriguez Hertz, M. Rodriguez Hertz, A. Tahzibi and R. Ures in [29, 31]. Using this argument, the authors proved the Pugh and Shub's Conjecture with two-dimensional center bundle in C^1 -topology.

Highly motivated by the Stably Ergodic Conjecture, our main result (Theorem A) of this paper provides a large class of conservative partially hyperbolic diffeomorphisms which can be C^1 approximated by stably ergodic systems. Unlike [17] or [31], these systems considered here are more general and the center dimension is not necessarily two.

2 Preliminaries

Given $f \in \text{Diff}_\mu^1(M)$, by Oseledec Theory ([24]), there is a μ -full invariant set $\mathcal{O} \subset M$ such that for every $x \in \mathcal{O}$ there exist a splitting (which is called *Oseledec splitting*)

$$T_x M = E_1(x) \oplus \cdots \oplus E_{k(x)}(x)$$

and real numbers (the Lypunov exponents of μ) $\chi_1(f, x) < \chi_2(f, x) < \cdots < \chi_{k(x)}(f, x)$ satisfying $Df(E_j(x)) = E_j(fx)$ and

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \ln \|Df^n v\| = \chi_j(f, x)$$

for every $v \in E_j(x) \setminus \{0\}$ and $j = 1, 2, \dots, k(x)$. In the following, by counting multiplicity, we also rewrite the Lyapunov exponents of μ as

$$\lambda_1(f, x) \leq \lambda_2(f, x) \leq \cdots \leq \lambda_d(f, x).$$

For $i = 1, 2, \dots, d$, define

$$LE_i(f) = \int_M (\lambda_1(f, x) + \cdots + \lambda_i(f, x)) d\mu.$$

It is obvious that the continuous points of the *Lyapunov map*

$$f \in \text{Diff}_\mu^1(M) \mapsto (LE_1(f), \dots, LE_{d-1}(f)) \in \mathbb{R}^{d-1}$$

is a residual set \mathcal{R}_0 of $f \in \text{Diff}_\mu^1(M)$.

For $f \in \text{PH}_\mu^1(M)$, the distributions E^u and E^s are integrable and their integrable manifolds form two transversal foliations of M , the *strongly stable* and *strongly unstable foliations* of M , which we denote by \mathcal{W}^u and \mathcal{W}^s respectively. For every $x \in M$ the leaves $\mathcal{W}^u(x)$ and $\mathcal{W}^s(x)$ of the foliations containing x are smooth immersed submanifolds in M called the *strong unstable* and *strong stable global manifolds* at x (see e.g. [14, 21]).

Two points $x, y \in M$ are called *accessible* if there are points $x = z_0, z_1, \dots, z_{l-1}, z_l = y, z_i \in M$ such that $z_i \in \mathcal{W}^u(z_{i-1})$ or $z_i \in \mathcal{W}^s(z_{i-1})$ for $i = 1, \dots, l$. A diffeomorphism f is called an *accessible diffeomorphism* if it has the accessibility property, i.e., any pare points $x, y \in M$ are accessible. f is *essentially accessible* if there are μ -full measure subset $M' \subset M$ such that any pare points $x, y \in M'$ are accessible. f is *stably accessible* if there is a C^1 neighborhood of f composed by accessible diffeomorphisms.

Accessibility is important to show the ergodicity of partially hyperbolic diffeomorphisms. In [18], D. Dolgopyat and A. Wilkinson proved that stable accessibility is C^1 dense. That is

Lemma 2.1. ([18]) *There is a C^1 open and dense set \mathcal{R}_1 in $\text{PH}_\mu^r(M)$ ($r \geq 1$) such that each $f \in \mathcal{R}_1$ is accessible.*

The following lemma can be find in [15].

Lemma 2.2. *Let $f \in \text{PH}_\mu^2(M)$ and f is accessible. Then almost every orbit is dense in M .*

An ergodic measure ν of $f \in \text{Diff}^r(M)$ is called *hyperbolic* if all the Lyapunov exponents of ν is not zero. If $r > 1$, for every point $x \in \mathcal{O}$, there are *Pesin's stable* and *unstable* manifolds which we denote by $W^s(x)$ and $W^u(x)$. If f is also partially hyperbolic, we have $\mathcal{W}^s(x) \subset W^s(x)$ and $\mathcal{W}^u(x) \subset W^u(x)$.

Given a diffeomorphism f and a f -invariant set $K \subset M$, a Df -invariant splitting $T_x M = E(x) \oplus F(x)$ ($x \in K$) of the tangent bundle over K is *l -dominated* if for any $x \in K$ one has

$$\frac{\|D_x f^l|_{E(x)}\|}{m(D_x f^l|_{F(x)})} < \frac{1}{2}$$

and the dimension of $E(x)$ is independent of $x \in K$. We denote the domination by $E \prec_l F$ and call $\dim(E)$ the *index* of the domination. A Df -invariant splitting $T_x M = E_1(x) \oplus \cdots \oplus E_k(x)$ ($x \in K$) of the tangent bundle over K is *l -dominated* if for any $i < j$, one has $E_i \prec_l E_j$ for some l .

Dominated splitting is unique, transverse and continuous. Moreover, the dominated splitting has some robust properties (see e.g. [12]). A dominated splitting $E_1 \oplus \cdots \oplus E_k$ is called the *finest dominated splitting* if there is no dominated splitting in each invariant bundle E_i for all $i = 1, \dots, k$. Moreover, a splitting is called the *robust finest dominated splitting* if the continuation of the splitting is the finest dominated splitting of the C^1 perturbation diffeomorphism.

For a hyperbolic periodic point P of f , we denote by $\text{ind}(P)$ the *index* of P , where the index of P refers to the dimension of the stable bundle of P . The *homoclinic class* of a hyperbolic saddle P of a diffeomorphism f , denoted by $H(P, f)$, is the closure of the transverse intersections of the invariant manifolds (stable and unstable ones) of the orbit of P .

Lemma 2.3. ([9]) *There exists a residual set \mathcal{R}_2 of $\text{Diff}_\mu^1(M)$ such that all diffeomorphisms in \mathcal{R}_2 is transitive. Moreover, M is the unique homoclinic class.*

Connecting Lemma was firstly proved by S. Hayashi ([20]) and was extended by L. Wen and Z. Xia in [33] (see also [4]). The following Connecting Lemma which established by C. Bonatti and S. Crovisier from the proof of Hayashi's will permit us to create intersections between stable and unstable manifolds.

Lemma 2.4. (Connecting Lemma [9]) *Let Q, P be hyperbolic periodic points of a C^r ($r \geq 1$) transitive diffeomorphism preserving a smooth measure μ . Then, there exists a C^1 -perturbation $g \in C^r$ preserving μ such that $W^s(P) \cap W^u(Q) \neq \emptyset$.*

Blender has been introduced firstly in [10], and it is a very useful tool to understand the dynamical properties such as the transitivity and ergodicity (e.g. see [10, 11, 30, 31]). There are several definitions of blender([10, 12, 23, 30]). The following definition comes from [11] and [30].

Definition 2.5. Let P, Q be hyperbolic periodic points of a diffeomorphism f with index i and $i + 1$ respectively. We say that f has a cu-blender of index i associated to (P, Q) if

(1) P is a partially hyperbolic periodic point of f such that Df is expanding on E^c and $\dim(E^c) = 1$;

(2) there is a small open set $\text{Bl}^u(P)$ such that every $(d - i)$ -strip well placed in $\text{Bl}^u(P)$ transversely intersects $W^s(P)$;

(3) $W^u(Q) \cap \text{Bl}^u(P)$ contains a vertical disk D through $\text{Bl}^u(P)$, i.e., D is a $(d - i - 1)$ -disk which is centered at a point in $\text{Bl}^u(P)$, the radius of D is much bigger than the radius of $\text{Bl}^u(P)$ and D is almost tangent to E^u ;

(4) this property is C^1 -robust.

Define a cs-blender in an analogous way by concerning f^{-1} .

We also will use the following two lemmas.

Lemma 2.6. (Theorem C of [6]) There is a residual set $\mathcal{R}_3 \subset \text{Diff}_\mu^1(M)$ such that for every $f \in \mathcal{R}_3$, every f -invariant Borel set $\Lambda \subset M$, and every $\eta > 0$, if $g \in \text{Diff}_\mu^1(M)$ is sufficiently close to f then there exists a g -invariant Borel set $\tilde{\Lambda}$ such that $\tilde{\Lambda} \subset B_\eta(\Lambda)$ and $\mu(\tilde{\Lambda} \Delta \Lambda) < \eta$, where $B_\eta(\Lambda) = \{x \in M : \rho(x, y) < \eta \text{ for some } y \in \Lambda\}$ and ρ is the distance on M induced by the Riemannian metric.

Lemma 2.7. ([5]) C^∞ diffeomorphisms are dense in the set of C^1 diffeomorphisms preserving μ .

3 Proof of Theorem 1

We first give a lemma which is important to the proof of Theorem A.

Lemma 3.1. There is a residual subset \mathcal{R}_4 of $\text{Diff}_\mu^1(M)$ such that for every $f \in \mathcal{R}_4$, M is the unique homoclinic class. Moreover, if there are two hyperbolic saddles of indices α and β , then M contains a dense subset of saddles of index τ for all $\tau \in [\alpha, \beta] \cap \mathbb{N}$.

Proof. This is the direct result of Lemma 2.3 and the conservative version of Theorem 1.1 of [1] (or see the Lemma 3.8 of [23]). \square

Now, we recall the criteria of ergodicity of [31]. Given a diffeomorphism f and a hyperbolic periodic point P , we define two invariant sets:

$$\Lambda^u(P) = \{x \in \mathcal{O} : W^s(P) \pitchfork W^u(x) \neq \emptyset\}, \quad \Lambda^s(P) = \{x \in \mathcal{O} : W^u(P) \pitchfork W^s(x) \neq \emptyset\},$$

where \mathcal{O} is the set of Oseledec regular points and W^s (W^u) is the Pesin global stable (unstable) manifold.

Lemma 3.2. (Theorem A of [31]) Let $f \in \text{Diff}_\mu^r(M)$ for $r > 1$. If $\mu(\Lambda^s) > 0$ and $\mu(\Lambda^u) > 0$ for some hyperbolic periodic point P , then

$$\Lambda(P) := \Lambda^s(P) \cap \Lambda^u(P) = \Lambda^s(P) = \Lambda^u(P) \pmod{0}$$

and f is ergodic on $\Lambda(P)$. Moreover, f is non-uniformly hyperbolic on $\Lambda(P)$.

Remark 3.3. *It is obvious that if $\Lambda^s(P) \cup \Lambda^u(P) = M \pmod{0}$, then f is ergodic respect to μ .*

In the following Lemma, we give a dense subset $\mathcal{D} \subset \text{PH}_\mu^1(M)$. In fact, to prove Theorem A, we only need to prove that the stable ergodicity can C^1 approximate to each system of a dense subset of \mathcal{P} .

Lemma 3.4. *There is a C^1 residual subset \mathcal{D} in $\text{PH}_\mu^1(M)$ such that for any $f \in \mathcal{D}$, we have*

- (1) *f is stably accessible and*
- (2) *there exists a robust finest dominated splitting of Df ,*

$$TM = E_1 \oplus E_2 \oplus \cdots \oplus E_k, \quad (3.1)$$

such that the Lyapunov exponents at x in E_i are equal for μ -a.e. $x \in M$ and all $i = 1, 2, \dots, k$.

Proof. By Lemma 2.2 of [23], there is an open and dense subset $\mathcal{D}_1 \subset \text{PH}_\mu^1(M)$ such that, for each $f \in \mathcal{D}_1$, Df has a robust finest dominated splitting $TM = E_1 \oplus E_2 \oplus \cdots \oplus E_k$. By Lemma 2.1 and 2.3, there is a residual set $\mathcal{D}_2 \subset \mathcal{D}_1 \subset \text{PH}_\mu^1(M)$ such that each $f \in \mathcal{D}_2$ is stably accessible and M is the unique homoclinic class. Set $\mathcal{D} = \mathcal{D}_2 \cap \mathcal{R}_3$, where \mathcal{R}_3 refers to Lemma 2.6. We shall prove that \mathcal{D} satisfies the lemma.

For any $1 \leq i < d$ and $l \in \mathbb{N}$, denote $D_i(f, l)$ by the set of points x such that there is a l -dominated splitting of index i along the orbit of $x \in M$. Then $D_i(f, l)$ is a compact invariant set. Set

$$\Gamma_i(f, l) = M \setminus D_i(f, l) \text{ and } \Gamma_i(f, \infty) = \bigcap_{l=1}^{\infty} \Gamma_i(f, l), \text{ for } i = 1, 2, \dots, d-1.$$

We shall show that, up to zero measure, either $\Gamma_i(f, \infty) = M$ or $D_i(f, l) = M$ for some l . In fact, if $\mu(D_i(f, l)) = 0$ for all l , then $\Gamma_i(f, \infty) = M \pmod{0}$. If $\mu(D_i(f, l)) > 0$ for some l , by Lemma 2.6 and 2.7, for any $\eta > 0$ there is $g \in \text{Diff}_\mu^2(M)$ which is C^1 close to f and a g -invariant Borel set $\tilde{\Lambda}$ such that $\tilde{\Lambda} \subset B_\eta(D_i(f, l))$ and $\mu(\tilde{\Lambda} \Delta D_i(f, l)) < \eta$. Since f is stably accessible and g is C^2 , Lemma 2.2 implies that $\tilde{\Lambda}$ is dense in M . So, $D_i(f, l)$ is η -dense in M . By the arbitrary of η , $D_i(f, l)$ is dense in M . So we have $D_i(f, l) = M$ since $D_i(f, l)$ is compact.

Noting that (3.1) is finest dominates splitting, $D_i(f, l) = M$ if and only if $i = \dim(E_1) + \cdots + \dim(E_j)$ for some $j = 1, \dots, k$. By [8], for μ -a.e. $x \in M$ and any $i = 1, 2, \dots, k$, the Lyapunov exponents in E_i are equal. \square

Proof of Theorem A. For any $f \in \text{PH}_\mu^1$, $r \geq 1$ and $\varepsilon > 0$, we set

$$\mathcal{U}^r(f, \varepsilon) = \{g \in \text{PH}_\mu^r : g \text{ is } \varepsilon\text{-}C^1\text{-close to } f\}.$$

To prove Theorem A, we only need to prove that for any $f \in \mathcal{D} \cap \mathcal{P} \cap \mathcal{R}_0$ and any $\varepsilon > 0$, there is $g \in \mathcal{U}^2(f, \varepsilon)$ such that g is stably ergodic.

Since $f \in \mathcal{P}$, there is an ergodic measure μ_1 such that $\lambda_{\mu_1}^+ \leq 0$, where $\lambda_{\mu_1}^+$ is the largest and smallest Lyapunov exponent of μ_1 in E_f^c . By Ergodic Closing Lemma ([4]), μ_1 -a.e. point is well closable. Then, for any $\epsilon_1 \leq \frac{\epsilon}{3}$, there are $f' \in \mathcal{U}^1(f, \frac{\epsilon_1}{2})$ and a periodic point P' of f' such that $\lambda_{s+c}(P') \leq \frac{\epsilon_1}{3}$. If $\lambda_{s+c}(P') < 0$, then P' is a hyperbolic periodic point with index $s+c$. Otherwise, using the conservative version of Frank's Lemma, one can get a new diffeomorphism $f'' \in \mathcal{U}^1(f', \frac{\epsilon_1}{2})$ which has the periodic point P'' with index $s+c$. Anyway, for the ϵ_1 , there is $f_1 \in \mathcal{U}^1(f, \epsilon_1)$ such that f_1 has a hyperbolic periodic point P_1 with index $s+c$.

Since f has a robustly non-hyperbolic center bundle, if we select ϵ_1 small enough, there is an ergodic measure ν of f_1 such that $\lambda_{\nu}^- \geq 0$, where λ_{ν}^- is the smallest Lyapunov exponent of ν in $E_{f_1}^c$. By the similar discussion as above, for any $\epsilon_2 \leq \epsilon_1$, there is $f_2 \in \mathcal{U}^1(f_1, \epsilon_2)$ such that f_2 has a hyperbolic periodic point Q with index s . Since P_1 is a hyperbolic periodic point of f_1 , the continuation P of P_1 is a hyperbolic point of f_2 and has the same index as the P_1 's if ϵ_2 is small enough.

That is to say, for any $\epsilon_1 > 0$, there is $f_2 \in \mathcal{U}^1(f, 2\epsilon_1)$ such that f_2 has two hyperbolic periodic points P and Q with indices $s+c$ and s respectively.

If $\epsilon_3 \in (0, \epsilon_2)$ is small enough, any $g \in \mathcal{U}^1(f_2, \epsilon_3)$ has two hyperbolic periodic points of indices s and $s+c$ by the hyperbolicity of P and Q . Moreover, by Lemma 3.1, any $g \in \mathcal{U}^1(f_2, \epsilon_3) \cap \mathcal{R}_4$ has a dense subset of saddles of index i for all $i \in \{s, s+1, \dots, s+c\}$. So, there is an open set $\mathcal{V}_0 \subset \mathcal{U}^1(f_2, \epsilon_3)$ such that each $g \in \mathcal{V}_0$ has a subset of hyperbolic periodic points of index i for all $i \in \{s, s+1, \dots, s+c\}$.

Lemma 3.5. *If there are two partially hyperbolic points P', Q' with indices $i, i+1$ and one-dimension center, then there exists a cu -blender of index i associated to (P, Q) and P, Q are homoclinic related to P', Q' respectively.*

Proof. This is the conservative version of subsection 4.1 of [11]. One also can see the construction in [30]. \square

Now we continue to prove the Theorem. Selecting a diffeomorphism $g_0 \in \mathcal{V}_0$, since g_0 has two saddles of indices $s+c-1$ and $s+c$, by Lemma 3.5 and the robust property of blender, there is an open subset $\mathcal{V}_1 \subset \mathcal{V}_0$, such that any $g \in \mathcal{V}_1$ has a cu -blender of index $s+c-1$. Selecting a diffeomorphism $g_1 \in \mathcal{V}_1$, since g_1 has two saddles of indices $s+c-2$ and $s+c-1$, by Lemma 3.5 and the robust property of blender again, there is an open subset $\mathcal{V}_2 \subset \mathcal{V}_1$, such that any $g \in \mathcal{V}_2$ has a cu -blender of index $s+c-2$. Noting that $\mathcal{V}_2 \subset \mathcal{V}_1$, g also has a cu -blender of index $s+c-1$. Inductively, we obtain open sets $\mathcal{V}_c \subset \dots \subset \mathcal{V}_1$ such that for any $1 \leq i \leq c$ and any $g \in \mathcal{V}_i$, g has i cu -blenders of indices $s+c-1, s+c-2, \dots, s+c-i$ respectively. Especially, for any $g \in \mathcal{V}_c$ and any $s \leq i < s+c$, there is cu -blender of index i associated to $(P_{i,g}, Q_{i+1,g})$.

To continue the proof, we need the following lemma.

Lemma 3.6. *If ε is small enough, then there is $g \in \mathcal{V}_c \cap \text{Diff}^2(M)$ such that g is stably ergodic.*

Proof. Firstly, if

$$\int_M \lambda_{s+c}(f, x) d\mu < 0 \text{ (or } \int_M \lambda_{s+1}(f, x) d\mu > 0),$$

then for ε small enough, we have

$$\int_M \lambda_{s+c}(g, x) d\mu < 0 \text{ (or } \int_M \lambda_{s+1}(g, x) d\mu > 0), \quad \forall g \in \mathcal{V}_c$$

since f is the continuous point of Lyapunov map. Then, by Theorem 4 (or section 1.8) of [15], each $g \in \mathcal{V}_c \cap \text{Diff}^2(M)$ is stably ergodic.

Otherwise, there are $1 \leq t \leq c$ and $1 \leq \kappa < t$ such that

$$\int_M \ln |\det(Df|_{E_f^{c\kappa}})| d\mu < 0 \text{ and } \int_M \ln |\det(Df|_{E_f^{c\kappa+1}})| d\mu > 0,$$

where $E_f^c = E^{c_1} \oplus \cdots \oplus E^{c_t}$ is robustly finest dominated splitting. Then,

$$\int_M \ln |\det(Dg|_{E_g^{c\kappa}})| d\mu < 0 \text{ and } \int_M \ln |\det(Dg|_{E_g^{c\kappa+1}})| d\mu > 0 \quad (3.2)$$

for any $g \in \mathcal{V}_c$ if ε is small enough.

We Choose a diffeomorphism $h \in \mathcal{V}_c \cap \mathcal{D} \cap \mathcal{R}_0$. By Lemma 3.4,

$$\int_M \ln |\det(Dh|_{E_h^{c\tau}})| d\mu = \dim(E_g^{c\tau}) \cdot \int_M \lambda_{s+\dots+c_\tau}(h, x) d\mu, \text{ for } \tau = \kappa, \kappa + 1.$$

Then (3.2) implies that

$$\int_M \lambda_{s+\dots+c_\kappa}(h, x) d\mu < 0 \text{ and } \int_M \lambda_{s+\dots+c_\kappa+1}(h, x) d\mu > 0.$$

Noting that h is the continuous point of Lyapunov map, we have

$$\int_M \lambda_{s+\dots+c_\kappa}(g, x) d\mu < 0 \text{ and } \int_M \lambda_{s+\dots+c_\kappa+1}(g, x) d\mu > 0. \quad (3.3)$$

for some small neighborhood \mathcal{V}'_c of h and any $g \in \mathcal{V}'_c$. On the other hand, by Lemma 2.4, there is an open set $\mathcal{V} \subset \mathcal{V}'_c$ such that for any $g \in \mathcal{V}$ and any $s < i \leq s + c$, $P_{i,g}$ is homoclinic related to $Q_{i,g}$. Taking $g \in \mathcal{V} \cap \text{Diff}^2(M)$, we shall prove g is ergodic (and hence is stably ergodic) in the following.

Set

$$\Lambda^+ = \{x \in \mathcal{O} : \lambda_{s+\dots+c_\kappa+1}(x) > 0\}, \quad \Lambda^- = \{x \in \mathcal{O} : \lambda_{s+\dots+c_\kappa}(x) < 0\}.$$

By (3.3), $\mu(\Lambda^+) \cdot \mu(\Lambda^-) > 0$. Moreover, we have $\mu(\Lambda^+ \cup \Lambda^-) = 1$ because of the domination $E^{c_\kappa} \prec E^{c_\kappa+1}$. If we have showed that $\Lambda^+ \subset \Lambda^u \pmod{0}$ and $\Lambda^- \subset \Lambda^s \pmod{0}$, then g is ergodic by Lemma 3.2 and Remark 3.3. We only prove the first part and the proof of the second part is similar.

In fact, we shall prove that for almost every point $x \in \Lambda^+$, there is $y \in \text{orb}(x)$ such that $y \in \Lambda^u$. Then, by the invariance of Λ^u , we have $x \in \Lambda^u$.

Since g is accessible, by Lemma 2.2, there is a μ -full measure set \mathcal{O}' such that $\text{orb}(x)$ is dense in M for every $x \in \mathcal{O}'$. Recall that g has a cu -blender of index $s+c-1$ associated to (P_{s+c-1}, Q_{s+c}) . So, given $x \in \Lambda^+ \cap \mathcal{O}'$, there is $y \in \text{orb}(x)$ such that $y \in \text{Bl}^u(P_{s+c-1})$. By the Pesin's Stable Manifold Theorem, y has unstable manifold $W^u(y)$ of dimension $c_{\kappa+1} + \dots + c_t + u$ and $W^u(y)$ is tangent to the bundle $E^{\kappa+1} \oplus \dots \oplus E^u$. Moreover, $W^u(y)$ should intersect transversely the stable manifold of P_{s+c-1} (which has index $s+c-1$) since the strong unstable manifold $W^u(y)$ has uniform size and $y \in \text{Bl}^u(P_{s+c-1})$. That is to say, $W^u(y) \cap W^s(P_{s+c-1}) \neq \emptyset$. Since P_{s+c-1} is homoclinic related to Q_{s+c-1} , we have $W^u(y) \cap W^s(Q_{s+c-1}) \neq \emptyset$ by λ -Lemma.

Claim. $W^u(y) \cap W^s(P_{s+c-2}) \neq \emptyset$.

Proof of Claim. By the definition of blender, $W^u(Q_{s+c-1}) \cap \text{Bl}^u(P_{s+c-2})$ contains a vertical disk D through $\text{Bl}^u(P_{s+c-2})$. On the other hand, since $W^u(y) \cap W^s(Q_{s+c-1}) \neq \emptyset$, by the λ -Lemma, $g^n(W^u(y))$ contains a $(u+1)$ -dimension manifold closing to $W^u(Q_{s+c-1})$ for n large enough. So $g^n(W^u(y))$ contains a $(u+1)$ -dimension disk which cross through $\text{Bl}^u(P_{s+c-2})$. Then we can conclude that $g^n(W^u(y))$ contains a $(u+2)$ -strip which is well placed in $\text{Bl}^u(P_{s+c-2})$ and thus $W^u(y) \cap W^s(P_{s+c-2}) \neq \emptyset$ by the definition of blender. \square

Similarly, by induction, we have $W^u(y) \cap W^s(P_j) \neq \emptyset$, where $j = s + c_1 + \dots + c_\kappa$. That is to say, $y \in \Lambda^u$ and we complete the proof of the Lemma. \square

We continue to prove Theorem A. By Lemma 3.6, there is stably ergodic $g \in \mathcal{V}_c$. Noting that $\mathcal{V} \cap \text{Diff}^2(M) \subset \mathcal{U}^2(f, \varepsilon)$, we complete the proof. \square

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